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Nonzero-temperature dynamics near quantum phase transition in the isotropic Lipkin–Meshkov–Glick model: an open quantum system approach

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Abstract
The dynamics of a central qubit system coupled to an isotropic ferromagnetic Lipkin–Meshkov–Glick spin bath at nonzero temperature is studied. We derive exactly the reduced density matrix and investigate the pairwise thermal entanglement, the coherence and the concurrence of the central spins. Through the behavior of these quantities, we show that at very low temperatures the dynamics is sensitive to the presence of the critical point of the environment.

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1. Introduction

Quantum phase transitions (QPTs) are associated with a qualitative change in the ground state of a many-body quantum system, at the absolute zero temperature, when some relevant parameter varies across its critical value [1]. Their manifestation in many experimental results on the cuprate superconductors and organic conductors stimulated much attention during the last decade. Recently, the relation between the entanglement and the quantum phase transitions has been the subject of many studies [1–10]. The critical behavior of the former was proposed as a tool for detecting the presence of QPTs in multi-spin systems. Most of the investigations have dealt with the zero-temperature dynamics near the critical point at which the transition occurs. However, it is believed that quantum phase transitions leave their fingerprints at temperatures close to the zero absolute. Generally speaking, at such low temperatures, the long-time collective dynamics of a quantum many-body system is investigated using the concepts of order parameters and quasiparticles which lead, however, to a semiclassical description of the dynamics [11]. Moreover, at nonzero temperatures, quantum correlations are suppressed by the thermal fluctuations: there exits a threshold temperature above which the thermal entanglement ceases to exist. Thus, a deep understanding of the dynamics of multi-spin systems at low temperatures is of theoretical and experimental significance.

The Lipkin–Meshkov–Glick (LMG) model [12–14], initially introduced in nuclear physics, has found many physical applications such as the Josephson effect and the two-mode Bose–Einstein condensate [15–17]. This model was extensively used to investigate the connection between the zero-temperature entanglement and QPTs [18–23]. The Hamiltonian of the isotropic LMG model with N spins subjected to a magnetic field of strength h is explicitly given by

$$H = \frac{g}{2N}\sum_{i<j}^{N}\left(\sigma_x^i\sigma_x^j + \sigma_y^i\sigma_y^j\right) + \frac{h}{2}\sum_i^N \sigma_z^i, \tag{1}$$

where g is the coupling constant and $\vec{\sigma}^i = 2\vec{S}^i$ is the Pauli operator corresponding to the particle labeled by i. The above Hamiltonian can be cast, up to an additive constant, into the form

$$H = \frac{g}{N}\left(\mathcal{J}^2 - \mathcal{J}_z^2\right) + h\mathcal{J}_z, \tag{2}$$

where $\vec{\mathcal{J}} = \sum_i^N \vec{S}^i$ is the total angular momentum of the multi-spin system. The standard basis of H is composed of the state vectors $|j, m\rangle$ common to $\mathcal{J}^2$ and $\mathcal{J}_z$ such that $0 \leqslant j \leqslant N/2$ and $-j \leqslant m \leqslant j$ (we set $\hbar = 1$). In the ferromagnetic case, i.e. $g < 0$, the ground state and the first exited state belong to the subspace $\mathbb{C}^{N+1}$ spanned by the eigenvectors $|N/2, m\rangle$. The model Hamiltonian displays a second-order phase transition at the critical point $|h_c| = -g$. Indeed, for $|h| > |h_c|$, the ground state is unique and is equal to the fully polarized state $|N/2, -\mathrm{sign}(h_c)N/2\rangle$ (symmetric phase), where $\mathrm{sign}(x)$ designates the sign of x. In contrast, in the domain $|h| < |h_c|$, the ground state depends on the coupling constant g (symmetry broken phase); its explicit form is given by $\left|N/2, I\left(\frac{hN}{2g}\right)\right\rangle$, where $I(x)$ denotes the round value of x.

In this paper, we apply the general formalism of open quantum systems to investigate the dynamics at low temperatures near the critical point of the isotropic LMG model. The idea consists of deriving the reduced density matrix of a central spin system which is coupled to a spin bath governed by the Hamiltonian (1). In section 2, we derive the one-qubit and two-qubit thermal reduced density matrix and investigate the pairwise thermal entanglement. In section 3, we study the time evolution of the coherence and the entanglement of, respectively, a single and a two spin $\frac{1}{2}$ particles coupled via Heisenberg or Ising interactions to the LMG spin bath. We end the paper with a short summary.

2. Thermal reduced density matrix

Let $\rho_N(0)$ denote the total density matrix of the multi-spin system whose dynamics is governed by the Hamiltonian (1). The state of any subsystem with m spins is fully described by its reduced density matrix, obtained by eliminating the degrees of freedom corresponding to the remaining $N - m$ particles. Note that H is invariant with respect to the exchange of sites; it follows that the reduced density matrix should be independent of the choice of the central particles. In this paper, we assume that our multi-spin system is in thermal equilibrium at arbitrary temperature T. The corresponding total density matrix is given by the Gibbs thermal state

$$\rho_N(0) = \frac{\exp(-H/T)}{\mathrm{tr}_N\{\exp(-H/T)\}}, \tag{3}$$

where the Boltzmann constant is set to 1 and $Z = \mathrm{tr}_N\{\exp(-H/T)\}$ is the partition function. Here, tr_N designates the trace with respect to the full set of the eigenvectors of H. In the following, we derive the reduced density matrix for both one and two central qubits. Without loss of generality we suppose that $g = -1$, and we only consider positive values of h since the spectrum of H is odd.

2.1. The one-particle reduced density matrix

First of all, it should be noted that the reduced density matrix can be obtained by directly calculating the mean values of the operators $\mathcal{J}^2$ and $\mathcal{J}_z$ with respect to the thermal state [24]. However, we shall proceed differently and use another method which allows us to investigate, in a straightforward manner, the time evolution of a central qubit coupled to the isotropic LMG bath (see the next section). Furthermore, one is usually seeking new techniques that lead to exact analytical results.

Let us choose one arbitrary particle, whose spin vector operator is denoted by $\vec{S}$, and call it central spin. The remaining $N-1$ particles can be viewed as a spin bath with a total angular momentum $\vec{J}$. At this stage it is useful to decompose the total spin vector of the full system as the sum of those corresponding to the central particle and the bath, namely,

$$\vec{\mathcal{J}} = \vec{S} + \vec{J}, \qquad \mathcal{J}_z = S_z + J_z. \tag{4}$$

Then, one can easily show that the isotropic Lipkin–Meshkov–Glick Hamiltonian can be rewritten in terms of the new spin operators as

$$H = \frac{g}{N}\left[J^2 - J_z^2 + \frac{1}{2}\right] + h(J_z + S_z) + \frac{g}{N}\left[S_+J_- + S_-J_+\right], \tag{5}$$

where $L_\pm = L_x \pm \mathrm{i}L_y$. Hence the full system is equivalent to a central qubit coupled to a spin bath through Heisenberg XY interactions. Similarly, the spin space of the composite system, $(\mathbb{C}^2)^{\otimes N}$, can be decomposed as

$$\mathbb{C}^2 \otimes (\mathbb{C}^2)^{\otimes N-1} = \mathbb{C}^2 \otimes \left[\bigoplus_{j}^{\frac{N-1}{2}} \nu(N-1, j)\mathbb{C}^{2j+1}\right], \tag{6}$$

where [25]

$$\nu(N, j) = \binom{N}{\frac{N}{2} - j} - \binom{N}{\frac{N}{2} - j - 1}. \tag{7}$$

The basis of the latter space is formed by the vectors $|k\rangle \otimes |j, m\rangle$, where $S_z|k\rangle = -\frac{(-1)^k}{2}|k\rangle$ ($k \in \{0, 1\}$), $J^2|j, m\rangle = j(j+1)|j, m\rangle$ and $J_z|j, m\rangle = m|j, m\rangle$. Note that in equation (6), the summation over j takes into account whether N is odd or even. Also, due to the last term on the right-hand side of equation (5), the Hamiltonian operator H is no longer diagonal in this new basis.

The method we adopt here is based on the fact that the operator $\varrho = Z\rho_N(0) = \exp[-\beta H]$ where $\beta = 1/T$ satisfies the following equation:

$$\frac{\partial}{\partial\beta}\varrho = -H\varrho. \tag{8}$$

In the standard basis of $\mathbb{C}^2$, the above operator can be written as $\varrho = \sum_{k,\ell} \varrho_{k\ell}|k\rangle\langle\ell|$. Consequently, equations (8) and (5) yield a set of four coupled first-order differential equations, namely,

$$\frac{\partial}{\partial\beta}\varrho_{11} = -\left[\frac{g}{N}\left(J^2 - J_z^2 + \frac{1}{2}\right) + h\left(J_z - \frac{1}{2}\right)\right]\varrho_{11} - \frac{g}{N}J_+\varrho_{21}, \tag{9}$$

$$\frac{\partial}{\partial\beta}\varrho_{21} = -\frac{g}{N}J_-\varrho_{11} - \left[\frac{g}{N}\left(J^2 - J_z^2 + \frac{1}{2}\right) + h\left(J_z + \frac{1}{2}\right)\right]\varrho_{21}, \tag{10}$$

$$\frac{\partial}{\partial\beta}\varrho_{22} = -\left[\frac{g}{N}\left(J^2 - J_z^2 + \frac{1}{2}\right) + h\left(J_z + \frac{1}{2}\right)\right]\varrho_{22} - \frac{g}{N}J_-\varrho_{12}, \tag{11}$$

$$\frac{\partial}{\partial\beta}\varrho_{12} = -\frac{g}{N}J_+\varrho_{22} - \left[\frac{g}{N}\left(J^2 - J_z^2 + \frac{1}{2}\right) + h\left(J_z - \frac{1}{2}\right)\right]\varrho_{12}. \tag{12}$$

The latter can be transformed into diagonal ones by introducing the following transformations [26]:

$$\varrho_{11} = \exp\left\{-\beta\left[\frac{g}{N}\left(J^2 - J_z^2 + \frac{1}{2}\right) + h\left(J_z - \frac{1}{2}\right)\right]\right\}V_{11}, \tag{13}$$

$$\varrho_{21} = J_-\exp\left\{-\beta\left[\frac{g}{N}\left(J^2 - J_z^2 + \frac{1}{2}\right) + h\left(J_z - \frac{1}{2}\right)\right]\right\}V_{21}, \tag{14}$$

$$\varrho_{22} = \exp\left\{-\beta\left[\frac{g}{N}\left(J^2 - J_z^2 + \frac{1}{2}\right) + h\left(J_z + \frac{1}{2}\right)\right]\right\}V_{22}, \tag{15}$$

$$\varrho_{12} = J_+\exp\left\{-\beta\left[\frac{g}{N}\left(J^2 - J_z^2 + \frac{1}{2}\right) + h\left(J_z + \frac{1}{2}\right)\right]\right\}V_{12}. \tag{16}$$

Using the commutation relations $[J_z, J_\pm] = \pm J_\pm$ and $\left[J_z^2, \pm\right] = \mp J_\pm(2J_z \pm 1)$, it can be shown that the operator variables V_{ij} satisfy

$$\frac{\partial}{\partial\beta}V_{11} = -\frac{g}{N}J_+J_-V_{21}, \qquad \frac{\partial}{\partial\beta}V_{21} = -\frac{g}{N}V_{11} - \frac{g}{N}(2J_z - 1)V_{21}, \tag{17}$$

$$\frac{\partial}{\partial\beta}V_{22} = -\frac{g}{N}J_-J_+V_{12}, \qquad \frac{\partial}{\partial\beta}V_{12} = -\frac{g}{N}V_{22} + \frac{g}{N}(2J_z + 1)V_{12}. \tag{18}$$

Combining equations (17) leads to the following second-order differential equation:

$$\ddot{V}_{21} + 2\frac{g}{N}\left(J_z - \frac{1}{2}\right)\dot{V}_{21} - \left(\frac{g}{N}\right)^2 J_+J_-V_{21} = 0. \tag{19}$$

It is worth mentioning that $\lim_{T\to\infty}\varrho = \lim_{\beta\to 0}\varrho = \mathbb{I}_N$, where $\mathbb{I}_N$ denotes the 2^N-dimensional unit matrix. Therefore, $V_{ii}(\beta = 0) = \mathbb{I}_{N-1}$ and $V_{ij}(\beta = 0) = 0$ for $i \neq j$. Taking into account the last conditions, it is easy to show that the general form of the solutions of equation (19) is given by

$$V_{21} = 2A\,\mathrm{e}^{-\frac{g\beta}{N}(J_z - \frac{1}{2})}\sinh\left[\frac{|g|\beta}{N}\sqrt{\left(J_z - \frac{1}{2}\right)^2 + J_+J_-}\right], \tag{20}$$

where A is a yet-to-be-determined diagonal operator. It is then sufficient to integrate the right-hand side of the first equation in (17) to obtain the exact form of V_{11}. Taking into account the values of V_{ij} at $\beta = 0$, one can find that $A = -\sqrt{(g/N)^2[(J_z - 1/2)^2 + J_+J_-]}/(2g)$, and thus by virtue of the transformations (13) and (14) we obtain

$$\varrho_{11} = \mathrm{e}^{-\beta G_1}\left\{\cosh\left[\frac{|g|\beta}{N}\sqrt{\left(J_z - \frac{1}{2}\right)^2 + J_+J_-}\right] + \frac{\mathrm{sign}(g)\left(J_z - \frac{1}{2}\right)}{\sqrt{\left(J_z - \frac{1}{2}\right)^2 + J_+J_-}}\sinh\left[\frac{|g|\beta}{N}\sqrt{\left(J_z - \frac{1}{2}\right)^2 + J_+J_-}\right]\right\}, \tag{21}$$

$$\varrho_{21} = \text{sign}(-g) J_- \, e^{-\beta G_1} \frac{1}{\sqrt{\left(J_z - \frac{1}{2}\right)^2 + J_+ J_-}} \sinh\left[\frac{|g|\beta}{N}\sqrt{\left(J_z - \frac{1}{2}\right)^2 + J_+ J_-}\right]\Bigg\}, \tag{22}$$

where $G_1 = \frac{g}{N}(J^2 - J_z^2 - 1/2) + (h + \frac{g}{N})\left(J_z - \frac{1}{2}\right)$ and $\text{sign}(g)$ designates the sign of the coupling constant g.

Similarly, it can be shown that V_{12} satisfies

$$\ddot{V}_{12} - 2\frac{g}{N}\left(J_z + \frac{1}{2}\right)\dot{V}_{12} - \left(\frac{g}{N}\right)^2 J_- J_+ V_{12} = 0. \tag{23}$$

Following the same method presented above, we find that

$$\varrho_{22} = e^{-\beta G_2}\Bigg\{\cosh\left[\frac{|g|\beta}{N}\sqrt{\left(J_z + \frac{1}{2}\right)^2 + J_- J_+}\right] + \frac{\text{sign}(-g)\left(J_z + \frac{1}{2}\right)}{\sqrt{\left(J_z + \frac{1}{2}\right)^2 + J_- J_+}} \sinh\left[\frac{|g|\beta}{N}\sqrt{\left(J_z + \frac{1}{2}\right)^2 + J_- J_+}\right]\Bigg\}, \tag{24}$$

$$\varrho_{12} = \text{sign}(-g) J_+ \, e^{-\beta G_2} \frac{1}{\sqrt{\left(J_z + \frac{1}{2}\right)^2 + J_- J_+}} \sinh\left[\frac{|g|\beta}{N}\sqrt{\left(J_z + \frac{1}{2}\right)^2 + J_- J_+}\right]\Bigg\}, \tag{25}$$

where $G_2 = \frac{g}{N}\left(J^2 - J_z^2 + 1/2\right) + \left(h - \frac{g}{N}\right)\left(J_z + \frac{1}{2}\right)$.

In order to obtain the reduced density matrix corresponding to the central spin-$\frac{1}{2}$ particle, we need to perform the trace in the space spanned by the common eigenvectors of J^2 and J_z. This task can be carried out with the help of the relation

$$\tilde{f} = \text{tr}_{N-1}\{f(J^2, J_z)\} = \sum_{j,m} \nu(N-1, j) f[j(j+1), m], \tag{26}$$

where f is some function of J^2 and J_z. Since the trace of the lowering and raising operators is identically zero, we can immediately infer that the reduced density matrix is diagonal in the standard basis of $\mathbb{C}^2$, namely,

$$\rho = \frac{1}{Z}\begin{pmatrix} \tilde{\varrho}_{11} & 0 \\ 0 & \tilde{\varrho}_{22} \end{pmatrix}, \tag{27}$$

where the elements $\tilde{\varrho}_{ii}$ are calculated using equation (26). It follows that the mean value of S_z, the purity and the von Neumann entropy corresponding to ρ are, respectively, given by $\langle S_z \rangle = \left(\frac{1}{Z}\left(\tilde{\varrho}_{22} - \tilde{\varrho}_{11}\right)\right)/2z$, $P = \text{tr}\,\rho^2 = \frac{1}{Z^2}\left(\tilde{\varrho}_{11}^2 + \tilde{\varrho}_{22}^2\right)$ and $S(\rho) = -(\tilde{\varrho}_{11}/Z)\ln(\tilde{\varrho}_{11}/Z) - (\tilde{\varrho}_{22}/Z)\ln(\tilde{\varrho}_{22}/Z)$.

Figures 1–3 display the variation of the above quantities as a function of the strength of the magnetic field at different values of the number of spins. We can see that $\langle S_z \rangle$ vanishes for $h = 0$ regardless of the values of N and T. This follows from the fact that, when h is equal to zero, the operator H reduces to Heisenberg XY Hamiltonian, which is invariant under rotations with respect to the z-direction. The above operator is clearly even function of J_z, which is also the case for the corresponding density matrix, $\rho_N(0)$: the thermal average of J_z is identically equal to zero.

From the above figures one can also see that, in the symmetry broken phase, starting from some value h_0 in the neighborhood of the critical point $h_c = 1$, the von Neumann entropy

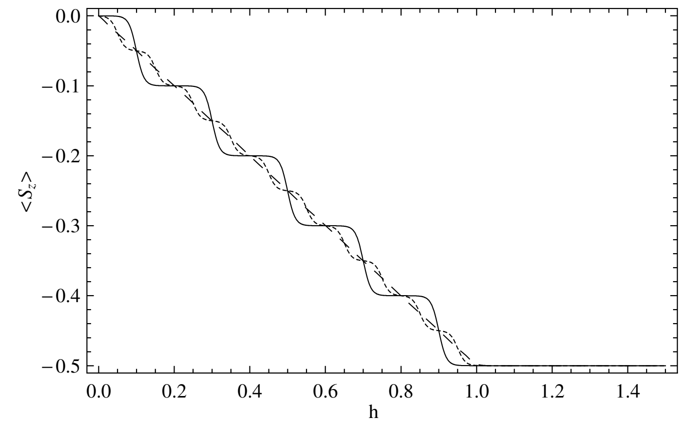

Figure 1. The dependence of the mean value of S_z on the strength of the magnetic field at different values of the number of spins: $N = 10$ (solid line), $N = 20$ (dashed line) and $N = 30$ (dotted line) with $T = 0.01$.

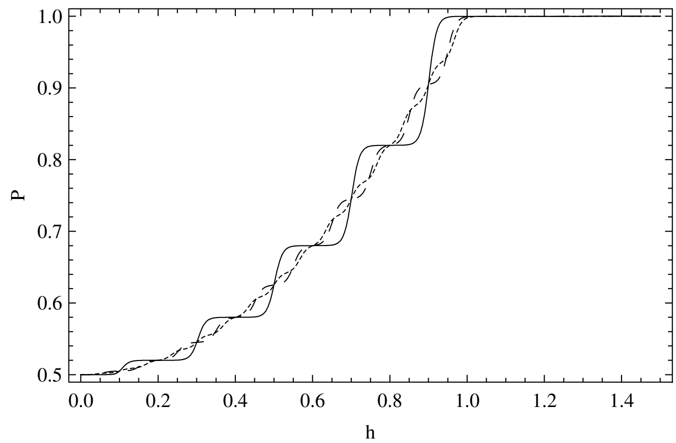

Figure 2. The dependence of the mean value of the purity of the reduced density matrix on the strength of the magnetic field at different values of the number of spins: $N = 10$ (solid line), $N = 20$ (dashed line) and $N = 30$ (dotted line) with $T = 0.01$.

vanishes whereas $\langle S_z \rangle$ and P become identically equal to -0.5 and 1, respectively. This means that all the spins are pointing in the direction of the magnetic field. Obviously, the above quantities maintain these values in the symmetric phase since the ground state of the spin system is equal to the fully polarized state vector $|N/2, -N/2\rangle$. Furthermore, it can be seen that the variation of $\langle S_z \rangle$, P and $S(\rho)$ is accompanied in the broken phase by some kind of oscillations which are more appreciable at small values of N. This can be explained by the dependence of the ground state $|N/2, -I(hN/2)\rangle$, which exhibits at low temperatures the largest statistical weight, on the strength of the magnetic field. Clearly, the quantity $I(hN/2)$ has the structure of a step function with respect to h; as T increases, the mean value of S_z slightly deviates from $-I(hN/2)/N$. A similar behavior is also observed for the purity and the von Neumann entropy. As we increase the number of spins and/or the temperature T, the

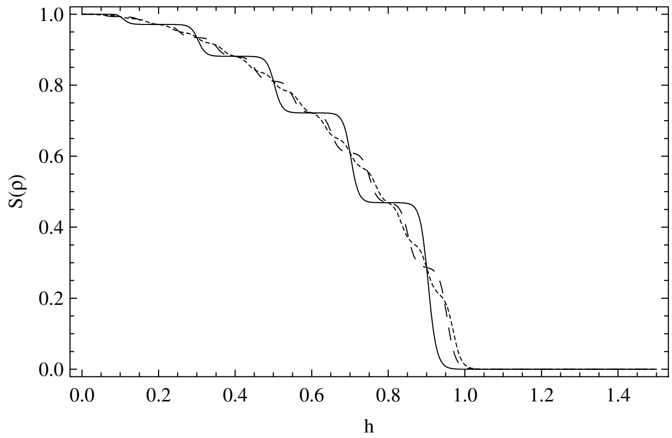

Figure 3. The von Neumann entropy as a function of the strength of the magnetic field at different values of the number of spins: $N = 10$ (solid line), $N = 20$ (dashed line) and $N = 30$ (dotted line) with $T = 0.01$.

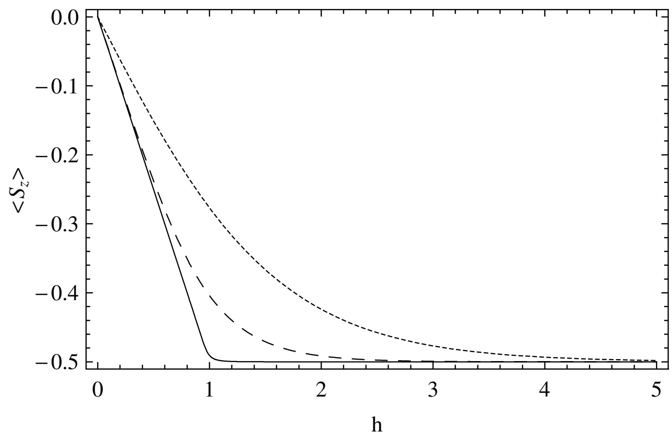

Figure 4. The mean value of S_z as a function of the strength of the magnetic field at different values of the temperature: $T = 0.1$ (solid line), $T = 0.4$ (dashed line) and $T = 0.8$ (dotted line) with $N = 300$.

oscillations completely disappear. Also, we observe that $h_0 \to h_c$ for $N \to \infty$ and $T \to 0$, as expected, since in this limit $I(hN/2)/N \approx h/2$. The behavior of the above quantities at large N is shown in figures 4–6.

2.2. *The two-particle reduced density matrix, pairwise thermal entanglement*

Next, consider entanglement properties of the isotropic Lipkin–Meshkov–Glick model at temperatures close to the zero absolute. The relevant quantity we shall look for is the two-spin reduced density matrix, ρ. A knowledge of the latter enables one to quantify the pairwise thermal entanglement between the pairs of spin-$\frac{1}{2}$ particles. The simplest measure we can use is the so-called concurrence, explicitly defined by $C(\rho) = \max\left\{0, 2\max[\sqrt{\lambda_i}] - \sum_{i=1}^{4}\sqrt{\lambda_i}\right\}$

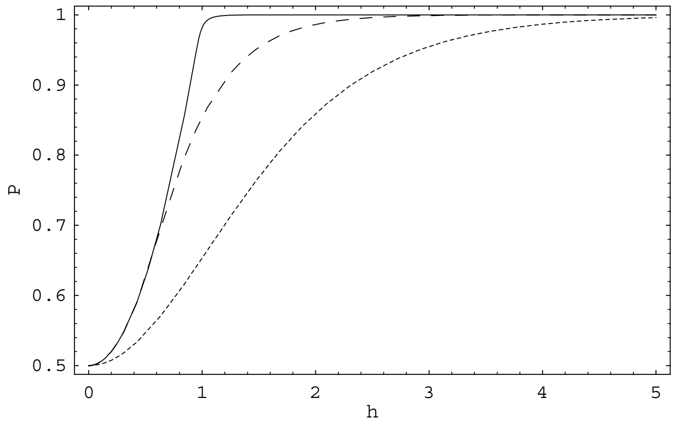

Figure 5. The dependence of the purity on the strength of the magnetic field at different values of the temperature: $T = 0.1$ (solid line), $T = 0.4$ (dashed line) and $T = 0.8$ (dotted line) with $N = 300$.

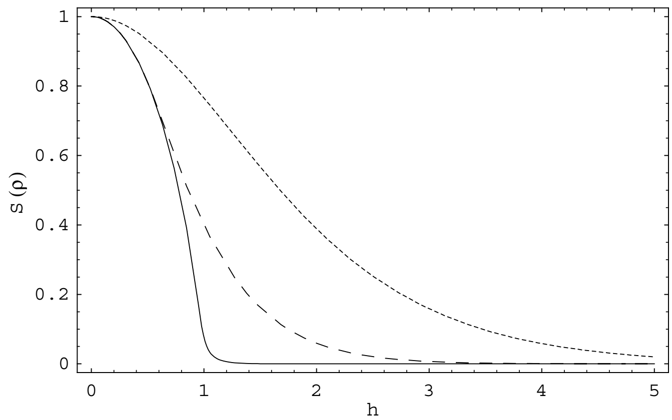

Figure 6. Variation of the von Neumann entropy with the strength of the magnetic field at different values of the temperature: $T = 0.1$ (solid line), $T = 0.4$ (dashed line) and $T = 0.8$ (dotted line) with $N = 300$.

[27], where λ_i are the eigenvalues of the operator $\rho(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$. It is worth mentioning that due to the invariance with respect to exchange of sites, the two-spin reduced density matrix in the space $\mathbb{C}^2 \otimes \mathbb{C}^2$ takes the form

$$\rho = \begin{pmatrix} a_- & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a_+ \end{pmatrix}, \tag{28}$$

where, $a_\pm, b$ and c are real numbers. The fact that c is real ensures that the reduced density matrix is diagonal in the space $\mathbb{C}^3 \oplus \mathbb{C} \equiv \mathbb{C}^2 \otimes \mathbb{C}^2$ spanned by the vectors $\{|1, -1\rangle, |1, 0\rangle, |1, 1\rangle, |0, 0\rangle\}$ [25]. The method presented in the previous subsection can also be applied here to derive the explicit form of ρ; the results are presented in the appendix.

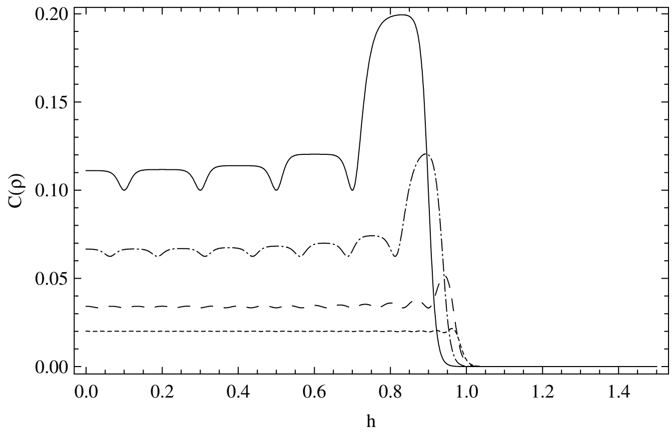

Figure 7. The pairwise thermal entanglement as a function of h at different values of N: $N = 10$ (dotted line), $N = 15$ (dot-dashed line), $N = 30$ (dashed line) and $N = 50$ (dotted line) with $T = 0.01$.

One can check their equivalence with the results of [24], where the elements of the density matrix (28) are shown to be explicitly given by

$$a_{\pm} = \frac{N^2 - 2N + 4\langle \mathcal{J}_z^2 \rangle \pm 4(N-1)\langle \mathcal{J}_z \rangle}{4N(N-1)}, \tag{29}$$

$$b = \frac{N^2 - 4\langle \mathcal{J}_z^2 \rangle}{4N(N-1)}, \tag{30}$$

$$c = \frac{\langle \mathcal{J}_+\mathcal{J}_- + \mathcal{J}_-\mathcal{J}_+ \rangle - N}{2N(N-1)}, \tag{31}$$

where the thermal average is defined as $\langle L \rangle \equiv \mathrm{tr}_N\{L\rho_N(0)\}$.

The main aim here is to investigate the pairwise thermal entanglement in the Lipkin–Meshkov–Glick model. From figures 7 and 8, we can see that, even at nonzero temperature, the concurrence is still sensitive to the phase of the system. Clearly, the above quantity strongly depends on both the temperature and the number of spins of the system. It turns out that, at sufficiently low T (N), there exists a threshold N_0 (T_0) above which the pairwise concurrence becomes identically equal to zero. The values of N_0 and T_0 depend, however, on the temperature and the number of spins, respectively. Moreover, the concurrence displays, in the broken phase, oscillations in the form of steps whose amplitudes increase with the increase of h. Within the latter phase, $C(\rho)$ also exhibits a peak which rapidly falls to zero in the neighborhood of the critical point h_c. At slightly higher T, we can see that the accompanying oscillations together with the peak disappear; in this case, the concurrence is a monotonic decreasing function of the strength of the magnetic field. As N increases, $C(\rho)$ decreases until it becomes practically independent of h in the symmetry broken phase. For sufficiently large h, the concurrence is obviously zero since the state of the system is, to a good approximation, equal to its fully polarized ground state. Finally, note that the behavior of the mean value of S_z, the purity and the von Neumann entropy is quite similar to that of the one-particle case.

At zero temperature, the derivative of the concurrence with respect to h is expected to display divergence at the critical point. However, for small N, even at zero temperature, the

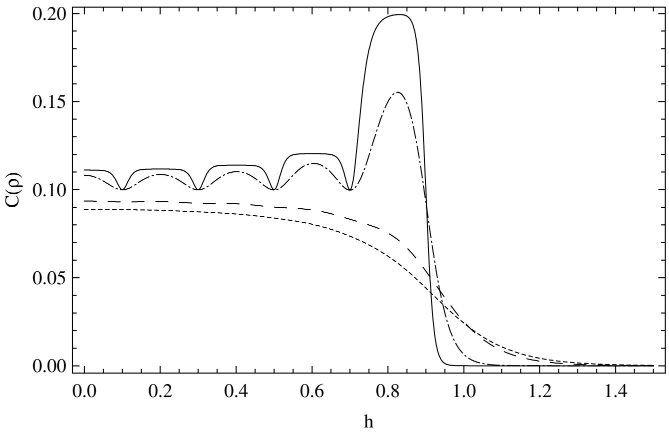

Figure 8. The pairwise thermal entanglement as a function of h at different values of T: $T = 0.01$ (solid line), $T = 0.03$ (dot-dashed line), $T = 0.08$ (dashed line) and $T = 0.1$ (dotted line) with $N = 10$.

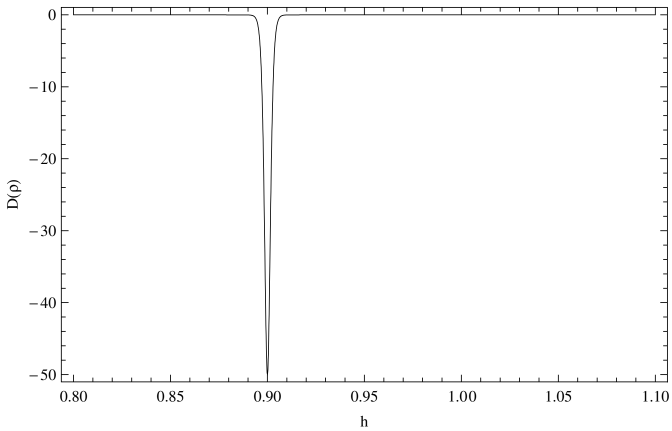

Figure 9. The derivative of concurrence as a function of h for $N = 10$ and $T = 0.001$.

concurrence vanishes at h_0 and not at $h_c = -g$. This is illustrated in figure 9 where the variation of $D(\rho) = \frac{\mathrm{d}C(\rho)}{\mathrm{d}h}$ as a function of h is shown for $N = 10$ and $T = 0.001$; note that the concurrence vanishes in the neighborhood of $h = 0.9$, which is exactly the value of h_0 when $T \to 0$. It is worth mentioning that the behavior of ground state entanglement (i.e. zero-temperature entanglement) of multi-spin system displaying quantum phase transition can be treated within the framework of density functional theory as shown in [28].

3. Coherence and concurrence dynamics

In this section we investigate the dynamics of the central spin system, assuming that its coupling constant to the bath, which we denote by α, is different from g. The former will be rescaled, as usual, by $\sqrt{N}$ to ensure an extensive free energy. The Hamiltonian operator describing the interaction between the two systems reads

$$H_I = \frac{\alpha}{\sqrt{N}}\left[S_+^0 J_- + S_-^0 J_+\right], \tag{32}$$

where $\vec{S}^0$ stands for the spin operator vector of the central system. Here, we use the notation $\vec{J}$ for the total spin instead of $\vec{\mathcal{J}}$ for convenience. In order to determine the exact analytical form of the time evolution operator, $\mathbf{U}(t)$, governing the unitary dynamics of the full system, we note that it satisfies the equation [29]

$$\mathrm{i}\frac{\mathrm{d}\mathbf{U}(t)}{\mathrm{d}t} = (H_0 + H_I + H)\mathbf{U}(t), \tag{33}$$

with $H_0 = hS_z^0$. Then the matrix elements of $\mathbf{U}(t)$ can be determined, for both one and two central spins, using the same method presented above. However, we shall not go through the details of the calculations since it is sufficient to make the replacement

$$\beta \to \mathrm{i}t, \tag{34}$$

and to take into account the fact that the coupling constants are different. In the case of a single central spin one can find that in $\mathbb{C}^2$

$$\begin{aligned} U_{11}(t) = \mathrm{e}^{-\mathrm{i}tG_1}&\left\{\cos\left[t\sqrt{\left(\left(\frac{g}{N}\right)\left(J_z - \frac{1}{2}\right)\right)^2 + \frac{\alpha^2}{N}J_+J_-}\right]\right.\\ &\left.+\frac{\mathrm{i}g/N\left(J_z-\frac{1}{2}\right)}{\sqrt{\left(\left(\frac{g}{N}\right)\left(J_z-\frac{1}{2}\right)\right)^2+\frac{\alpha^2}{N}J_+J_-}}\sin\left[t\sqrt{\left(\left(\frac{g}{N}\right)\left(J_z - \frac{1}{2}\right)\right)^2 + \frac{\alpha^2}{N}J_+J_-}\right]\right\}, \end{aligned} \tag{35}$$

$$\begin{aligned} U_{22}(t) = \mathrm{e}^{-\mathrm{i}tG_2}&\left\{\cos\left[t\sqrt{\left(\left(\frac{g}{N}\right)\left(J_z + \frac{1}{2}\right)\right)^2 + \frac{\alpha^2}{N}J_-J_+}\right]\right.\\ &\left.-\frac{\mathrm{i}g/N\left(J_z+\frac{1}{2}\right)}{\sqrt{\left(\left(\frac{g}{N}\right)\left(J_z+\frac{1}{2}\right)\right)^2+\frac{\alpha^2}{N}J_-J_+}}\sin\left[t\sqrt{\left(\left(\frac{g}{N}\right)\left(J_z + \frac{1}{2}\right)\right)^2 + \frac{\alpha^2}{N}J_-J_+}\right]\right\}, \end{aligned} \tag{36}$$

$$U_{12}(t) = J_+\frac{-\mathrm{i}(\alpha/\sqrt{N})\,\mathrm{e}^{-\mathrm{i}G_2}}{\sqrt{\left[\frac{g}{N}\left(J_z+\frac{1}{2}\right)\right]^2+\frac{\alpha^2}{N}J_-J_+}}\sin\left\{t\sqrt{\left[\frac{g}{N}\left(J_z+\frac{1}{2}\right)\right]^2+\frac{\alpha^2}{N}J_-J_+}\right\}, \tag{37}$$

$$U_{21}(t) = J_-\frac{-\mathrm{i}(\alpha/\sqrt{N})\,\mathrm{e}^{-\mathrm{i}G_1}}{\sqrt{\left[\frac{g}{N}\left(J_z-\frac{1}{2}\right)\right]^2+\frac{\alpha^2}{N}J_+J_-}}\sin\left\{t\sqrt{\left[\frac{g}{N}(J_z-\frac{1}{2})\right]^2+\frac{\alpha^2}{N}J_+J_-}\right\}. \tag{38}$$

The coherence of the central system which is assumed to be initially decoupled from the bath is then given by

$$S_-^0(t) = \frac{1}{Z}\mathrm{tr}_N\{\mathbf{U}(t)(S_-^0(0)\otimes \mathrm{e}^{-\beta H})\mathbf{U}^\dagger(t)\} = \Phi(t)S_-^0(0), \tag{39}$$

where

$$\Phi(t) = \frac{1}{Z}\mathrm{tr}_N\left[U_{11}(t)\mathrm{e}^{-\beta H}U_{22}^\dagger(t)\right]. \tag{40}$$

In the case of the two-qubit central system, we only consider the evolution in time of the maximally entangled state $|1,0\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$; other cases can be treated in exactly the same manner. It can be shown that the time-dependent reduced density matrix corresponding to the above initial state is diagonal in $\mathbb{C}^3$, with the matrix elements [25] $\rho_{\ell\ell}(t) = \mathrm{tr}_N[U_{\ell 2}\rho_N(0)U_{\ell 2}^\dagger]$ where

$$U_{22}(t) = \mathrm{e}^{-\mathrm{i}t[g/N(J^2-J_z^2)+hJ_z]}\sum_{k=1}^{3}\frac{\mathrm{e}^{r_k t}}{H_k}\left[\left(\frac{g}{N}\right)^2\left(1-4J_z^2\right)+2\mathrm{i}\frac{g}{N}r_k - r_k^2\right], \tag{41}$$

$$U_{12}(t) = -\frac{\sqrt{2}\alpha}{\sqrt{N}}J_+\,\mathrm{e}^{-\mathrm{i}t[g/N(J^2-J_z^2)+hJ_z]}\sum_{k=1}^{3}\frac{\mathrm{e}^{r_k t}}{H_k}\left[\left(\frac{g}{N}\right)\left(1-2J_z^2\right)+\mathrm{i}r_k\right], \tag{42}$$

$$U_{32}(t) = -\frac{\sqrt{2}\alpha}{\sqrt{N}}J_-\,\mathrm{e}^{-\mathrm{i}t[g/N(J^2-J_z^2)+hJ_z]}\sum_{k=1}^{3}\frac{\mathrm{e}^{r_k t}}{H_k}\left[\left(\frac{g}{N}\right)\left(1+2J_z^2\right)-\mathrm{i}r_k\right]. \tag{43}$$

Here, $H_k = \left(\frac{g}{N}\right)^2\left(1-4J_z^2\right) - 4\alpha^2/N(J_+J_- - J_z) + 4\mathrm{i}\frac{g}{N}r_k - 3r_k^2$; the quantities r_k are the solutions of the equation

$$r^3 - 2g/Nr^2 + \left[4\left(\frac{g}{N}\right)^2\left(4J_z^2-1\right)+4\frac{\alpha^2}{N}(J_+J_- - J_z)\right]r + 4\mathrm{i}\alpha^2(g/N^3)\left(J_z - 2J_z^2 - J_+J_-\right) = 0. \tag{44}$$

They are explicitly given by

$$r_1 = \frac{1}{3R}[2\mathrm{i}Rg/N - (K - R^2)], \tag{45}$$

$$r_2 = \frac{1}{6R}[4\mathrm{i}Rg/N + (1+\mathrm{i}\sqrt{3})(K + \mathrm{i}R^2)], \tag{46}$$

$$r_3 = \frac{1}{6R}[4\mathrm{i}Rg/N + (K - R^2) - \mathrm{i}\sqrt{3}(K + R^2)], \tag{47}$$

where

$$K = 12(\alpha^2/N)(J_+J_- - J_z) + (g/N)^2\left(1+12J_z^2\right), \tag{48}$$

and

$$R = \left[\mathrm{i}Qg/N + \tfrac{1}{2}\sqrt{4K^3 - 4(g/N)^2Q^2}\right]^{1/3}, \tag{49}$$

with

$$Q = (g/N)^2\left(-1+36J_z^2\right) - 18(\alpha^2/N)[J_+J_- + J_z(-1+6J_z)]. \tag{50}$$

The concurrence corresponding to the sate $|1,0\rangle$ is simply given by [25, 30] $C(t) = \max\{0, 2\max[\rho_{22}, \sqrt{\rho_{11}\rho_{33}}] - \rho_{22} - 2\sqrt{\rho_{11}\rho_{33}}\}$. The evolution in time of both $C(t)$ and the absolute value of $\Phi(t)$ is shown in figures 10 and 11. Clearly, the behavior of the above quantities depends on the phase of the system even though the temperature is different from zero. This actually becomes more clear as N increases in contrast with the thermal pairwise entanglement which exists only for small values of N. The change of the behavior of the concurrence and the coherence is related to the change of the ground state of the bath at the critical point. Once again we recall that the ground state is characterized by the largest

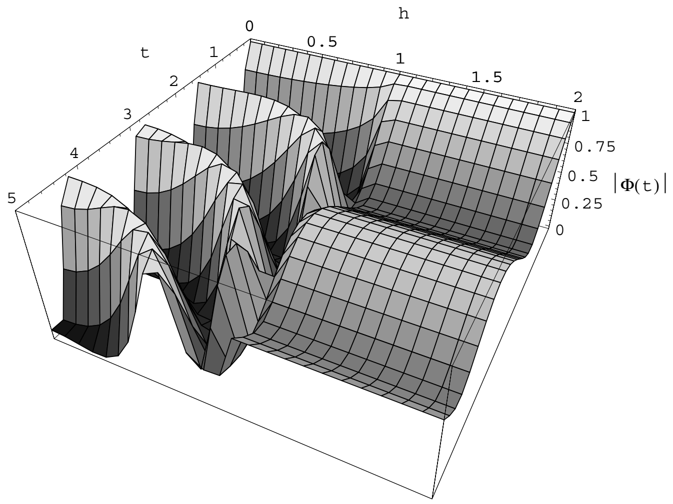

Figure 10. Time dependence of $|\Phi(t)|$ for different values of h with $N = 100$ and $T = 0.01$. The time variable is given in units of α.

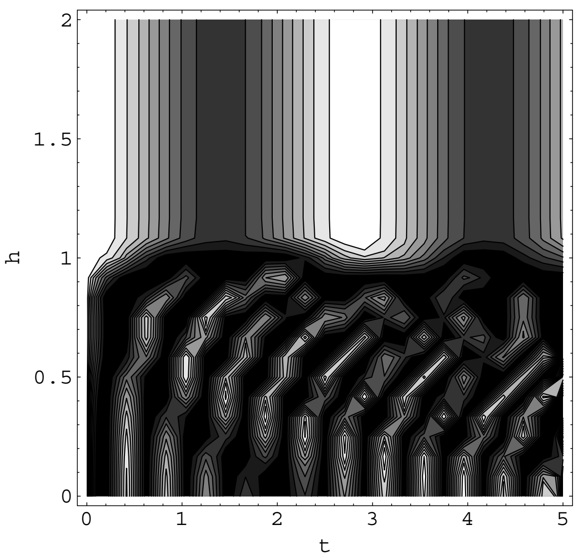

Figure 11. Contour plot showing the time dependence of $C(t)$ for different values of h with $N = 100$ and $T = 0.01$. The time variable is given in units of $\alpha/\sqrt{2}$.

statistical weight at low temperatures, which means that any perturbation of the latter state affects the time evolution of the central spins. It can be checked that at high temperatures the behavior of the reduced density matrix is exactly the same in both phases.

So far we have only considered Heisenberg XY interactions between the central system and the bath. Let us briefly investigate the case where the couplings are of Ising type. The corresponding interaction Hamiltonian operator is given by $H_I = \frac{\lambda}{\sqrt{N}} S_z^0 J_z$, where λ is the coupling constant. One can easily see that the Lipkin–Meshkov–Glick Hamiltonian (1)

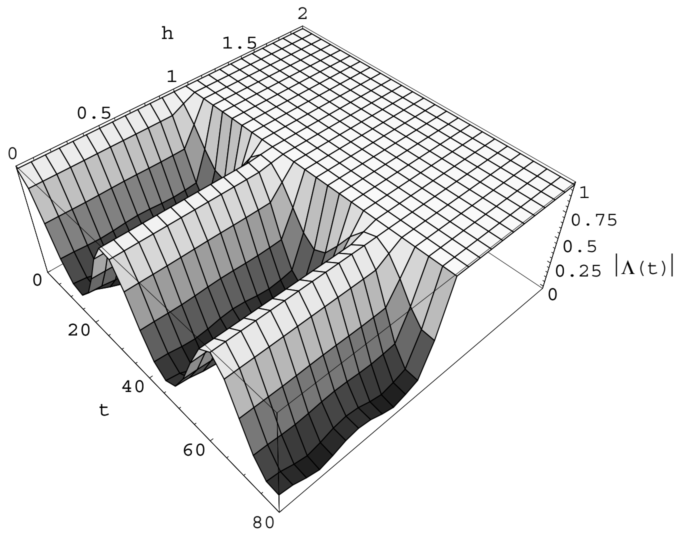

Figure 12. Time dependence of $|\Lambda(t)|$ at different values of h with $N = 100$ and $T = 0.01$; the time variable is given in units of λ.

commutes with H_I, that is $[H, H_I] = 0$. Therefore, in the case of a single central spin, the coherence is proportional to the function

$$\Lambda(t) = \frac{1}{Z}\sum_{j,m}\nu(N,j)\exp\left\{2\mathrm{i}h\,t - \beta[g/N(j(j+1)-m^2)+h\,m] + \frac{\mathrm{i}\lambda t}{\sqrt{N}}m\right\} \qquad (51)$$

whose dependence on the time and the strength of the magnetic field is illustrated in figure 12. This reveals that, at low temperatures, the absolute value of $\Lambda(t)$ is equal to 1 in the symmetric phase independently of the values of h. In the case of two central spins, the bell state $|1,0\rangle$ is found to be decoherence free: its concurrence does not evolve in time. However, the behavior of the concurrence corresponding to the maximally entangled state $\frac{1}{\sqrt{2}}(|1,1\rangle + |1,-1\rangle)$ is identical to that of $\Lambda(t)$, except that it decays twice faster than the above function. Once again, we find that the dynamics of the central system depends on the phase of the bath. As a final remark, note that the sudden change of the concurrence at the critical point above which it vanishes is quite similar to entanglement sudden death [31, 32]. One should not take this comparison too seriously since entanglement sudden death corresponds to the time dependence of entanglement. In our case, however, the parameter that controls the variation of entanglement is the strength of the magnetic field, externally applied to the spin bath. What really matters is the difference in the behavior of the dynamics in both phases rather than the vanishing of the entanglement itself.

4. Summary

In summary, we have investigated the nonzero-temperature dynamics of one and two central qubits coupled to an isotropic Lipkin–Meshkov–Glick bath near its critical point. We showed that the reduced density matrix of the central spin system can be exactly derived using an operator technique that makes use of the underlying symmetries of the model Hamiltonian. It is found that, at sufficiently low temperatures, the dynamics is sensitive to the phase of the bath. This is simply due to the fact that the main contribution to the thermal state of the bath comes from its ground state. For small values of the number of spins, the pairwise thermal entanglement clearly signals the existence of the critical point at which the transition occurs. However, above some threshold values of both the temperature and the number of spins within the bath, the pairwise thermal entanglement ceases to exist. This turns out to be not the case when the central spin system is not a part of the bath, i.e. its coupling constant is different form those of bath spins; here we find that the differences between the behavior of the concurrence within the two possible phases of the bath become more clear at large values of the number of spins.

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Appendix

In the two-spin case, it can be shown that in $\mathbb{C}^3 \oplus \mathbb{C}$ the first two diagonal elements of ϱ read

$$\begin{aligned}\varrho_{11} = \frac{2}{M^4 - M^2} \exp\left\{-\beta\left[\frac{g}{N}\left(J^2 - 3J_z^2\right) + h(J_z - 1)\right]\right\} \\ \times \left\{\left[J_+J_- + 4(J_+J_-)^2 - 2J_z - 12J_zJ_+J_- + 10J_z^2 + 12J_+J_-J_z^2 - 16J_z^3 + 8J_z^4\right]\right. \\ \times \cosh(g\beta M/N) - M\sinh(g\beta M/N)\left[3J_+J_- - 2J_z - 4J_zJ_+J_- + 6J_z^2 - 4J_z^3\right] \\ \left. + \left[J_+J_- + 4(J_+J_-)^2 - 4J_+J_-J_z + 4J_+J_-J_z^2\right]\mathrm{e}^{\frac{g\beta}{N}}\right\},\end{aligned} \tag{A.1}$$

$$\begin{aligned}\varrho_{22} = \frac{4\,\mathrm{e}^{\frac{g\beta}{N}}}{M^4 - M^2}\left\{\left[J_+J_- + 4(J_+J_-)^2 - J_z - 8J_+J_-J_z + 4J_z^2J_+J_- + 4J_z^2(1 - J_z)\right]\right. \\ \times \cosh(g\beta M/N) - M\sinh(g\beta M/N)\left[J_+J_- - J_z + 2J_z^2\right] \\ \left. + \mathrm{e}^{\frac{g\beta}{N}}\left[4J_+J_-J_z^2 + J_z^2 + 4J_z^3(J_z - 1)\right]\right\}\end{aligned} \tag{A.2}$$

where $M = \sqrt{1 - 4J_z + 4J_z^2 + 4J_+J_-}$. Due to the symmetry, the explicit form of the matrix element ϱ_{33} can be obtained from the expression of ϱ_{11} by simply making the substitution $h \to -h$. Moreover, since $J_\pm|0, 0\rangle \equiv 0$, then the fourth diagonal element corresponding to $\mathbb{C}$ is simply given by

$$\varrho_{44} = \exp\left\{-\beta\left[\frac{g}{N}\left(J^2 - J_z^2\right) + hJ_z\right]\right\}. \tag{A.3}$$

Finally, the elements of the two-spin reduced density matrix are given by

$$\rho_{ii} = \frac{1}{Z}\sum_{j,m} \nu(N - 2, j)\langle j, m|\varrho_{ii}|j, m\rangle. \tag{A.4}$$

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